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BY

GEORGE PÓLYA

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EDITORIAL NOTE

The two papers on exact sequential analysis, by Professor George Pólya and by Mrs. Julia Robinson, were submitted for publication early in the summer of 1946. At that time it was uncertain whether these papers should be published in the present series or in some other form, and this caused a delay in publication which is deeply regretted. The September issue of the *Annals of Mathematical Statistics*, Vol. XVII (1946), presented a paper by M. A. Girshick, "Contributions to the Theory of Sequential Analysis," anticipating some results of Professor Pólya and of Mrs. Robinson. The papers by these two authors are published now as they were submitted because of the difference in method and because of the numerical results, which seem to be important.

The Editors

EXACT FORMULAS IN THE SEQUENTIAL ANALYSIS OF ATTRIBUTES

BY

GEORGE PÓLYA

THIS PAPER reproduces, except for small changes, a talk given on April 9, 1946, in the Seminar of the Statistical Laboratory of the Department of Mathematics of the University of California, Berkeley. My talk was suggested by the preceding report of Miss E. L. Scott in the same seminar on the same subject. I am much indebted to Miss Scott for various suggestions and for the first numerical application of the following formulas; to Mrs. Julia Robinson for the more extended computations, of which a first account follows this paper; and to Professor Neyman for his hospitality and for his support of the computational work.

There is no need of commenting on the importance of the general idea of sequential tests of statistical hypotheses, or on the application of this idea to inspection plans for sampling attributes. (See [3], especially pp. 159–164, and [1], especially Sec. 2.) Wald found remarkable approximate formulas for the probability of accepting the lot and the average sample size necessary for a decision. (See [3] (5.11) and (5.13) or the equivalent [1] (2.301); and [3] (5.14) or the equivalent [1] (2.501).) The question about the degree of this approximation seems to remain open.

The aim of the following simple developments is to show that exact formulas can be obtained whenever the "slope" s is a rational number, and that these formulas are accessible to numerical computation with a reasonable amount of labor in certain cases (when neither the denominator of s, nor the sum $h_1 + h_2$ is large). I shall present the necessary developments without supposing any special preliminary knowledge. I shall concentrate on making clear the method of computation and refrain from comments on various perspectives, other applications, and possible generalizations.

It may be worth observing that in one case Wald's formulas are not only approximate, but *exact: when* $s = 1/2$, *and* $2h_1$ *and* $2h_2$ *are integers*. In this case, however, the problem is not new, but coincides with the traditional problem of the "duration of play" (or "la ruine du joueur"). This remark was the origin of the following developments, which use traditional tools of the calculus of probability.

1. **Geometric language.**[1] In the following, the word *point* refers only to points of the plane whose rectangular coördinates x and y are non-negative integers. The points $(x+1, y)$ and $(x, y+1)$ *follow immediately* the point (x,y). The point $(x+1, y)$ follows (x,y) in the x-direction, $(x, y+1)$ follows (x,y) in the y-direction. The points (x_0,y_0), (x_1,y_1), (x_2,y_2), $\cdots$, (x_l,y_l) constitute a *path* if $(x_{\lambda+1}, y_{\lambda+1})$ follows immediately (x_λ,y_λ), for $\lambda = 0, 1, 2, \cdots, l-1$. This path passes through the points (x_1,y_1), (x_2,y_2), $\cdots$, (x_{l-1}, y_{l-1}), its initial point is (x_0,y_0), its end point (x_l,y_l), and its length l.

[1] The following terminology is similar to, but in various points different from, that adopted in [2].

We are given three positive numbers, h_1, h_2, and s. We suppose that $s < 1$. The points (x,y) satisfying the inequality

$$-h_1 + s(x+y) < x < h_2 + s(x+y) \tag{1.1}$$

constitute the *strip* S. We consider only the case in which any point (x,y) of S can be attained as end point of a path which starts from (0,0) and *passes only through points of* S. Let $K(x,y)$ denote the number of all different paths of this kind. Our first object is to compute the number $K(x,y)$. In order to do so, however, we need a little more geometric preparation.

We say that a point belongs to the *active boundary* of S if it does not belong to S but is followed immediately by a point of S. We say that a point belongs to the *passive boundary* of S if it does not belong to S but *follows* immediately a point of S. We write AB and PB for active and passive boundary, respectively. The passive boundary consists of two parts, the *passive boundary in the* x-*direction* and the *passive boundary in the* y-*direction*, written shortly PBX and PBY, respectively. A point of PBX follows a certain point of S in the x-direction, a point of PBY follows a certain point of S in the y-direction. We have in each case a different inequality:

$$x \geqq h_2 + s(x+y) \quad \text{if} \quad (x,y) \quad \text{in} \quad PBX \tag{1.2}$$

$$x \leqq -h_1 + s(x+y) \quad \text{if} \quad (x,y) \quad \text{in} \quad PBY \tag{1.3}$$

Therefore, PBX and PBY have no common points, but their union, the whole passive boundary, may have common points with the active boundary.

In what follows, we restrict ourselves to the case in which s *is a rational number.* In this case, there are two positive integers, a and b, without common divisor, such that

$$\frac{s}{1-s} = \frac{a}{b} \tag{1.4}$$

therefore

$$s = \frac{a}{a+b}, \qquad 1 - s = \frac{b}{a+b} \tag{1.5}$$

and we can rewrite (1.1) in the form

$$-(a+b)h_1 < bx - ay < (a+b)h_2 . \tag{1.6}$$

Since a, b, x, y are integers, the condition expressed by (1.6) does not change at all if we *replace* (a + b)h_1 *by the next greater integer.* We shall do so, and we shall treat $(a+b)h_2$ similarly. If $(a+b)h_2$ was an integer from the outset, it remains unchanged, and the same holds for $(a+b)h_1$. Thus, in what follows, h_1 and h_2 denote rational numbers ($(a+b)h_1$ and $(a+b)h_2$ integers) unless the contrary is explicitly stated.

If the point (x,y) satisfies the inequality (1.6), the point $(x+a, y+b)$ satisfies

it too. Therefore, if s is rational, the strip S can be decomposed into an infinity of *segments* $S_0, S_1, S_2, \cdots$ without common points

$$S = S_0 + S_1 + S_2 + S_3 + \cdots \tag{1.7}$$

so that $S_1, S_2, S_3, \cdots$ are congruent. More explicitly: If (x,y) ranges over the points of S_n, the point $(x + a, y + b)$ describes S_{n+1} $(n = 1, 2, 3, \cdots)$. The whole strip S appears in (1.7) as decomposed into an infinite periodic part, $S_1 + S_2 + S_3 + \cdots$, and an initial nonperiodic part, S_0.

Let us consider the example in which

$$s = \frac{1}{4}, \quad h_1 = \frac{3}{2}, \quad h_2 = \frac{3}{2}. \tag{1.8}$$

We have therefore

$$a = 1, \qquad b = 3, \qquad s = \frac{1}{1+3}. \tag{1.9}$$

This example is illustrated by figures 1, 2, and 3. In these figures, the x-axis points downward and the y-axis to the right.[2] The points of the strip S appear in figure 1 as corners of an assemblage of squares. In figure 2, the number $K(x,y)$ marks the

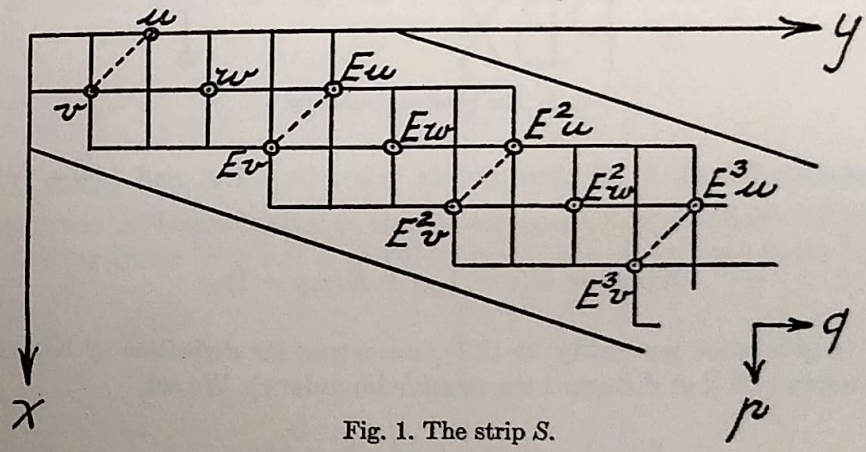

Fig. 1. The strip S.

1	1	1	1	1	1	0	0	0				
1	2	3	4	5	6	6	6	6	0	0	0	
0	2	5	9	14	20	26	32	38	38	38	38	0
			0	14	34	60	92	130	168	206	244	244
						0	92	222	390	596	840	
									0	596		

Fig. 2. Table of $K(x,y)$, which is 0 on the active boundary.

[2] This is convenient in numerical computation; the numbers $K(x,y)$ are arranged as the binomial coefficients usually are in Pascal's triangle. As a mnemotechnic rule we may remember: defectives go down, right items to the right.

place of the point (x,y) and 0 a point of the active boundary. In figure 3, the points of the passive boundary are marked with little circles. A point of PBX is connected with the point of S that it follows by a stroke parallel to the x-axis, and the points of PBY are similarly related to the y-axis. Figure 1 shows the segments S_0, S_1, S_2, · · · separated by dotted lines; S_0 contains three points; S_1, S_2, S_3, · · · contain eleven points each.

In computing $K(x,y)$ and solving our other problems, we shall discuss the special example characterized by (1.8) and illustrated by figures 1, 2, and 3, but we shall discuss it so that the general idea becomes clearly visible.

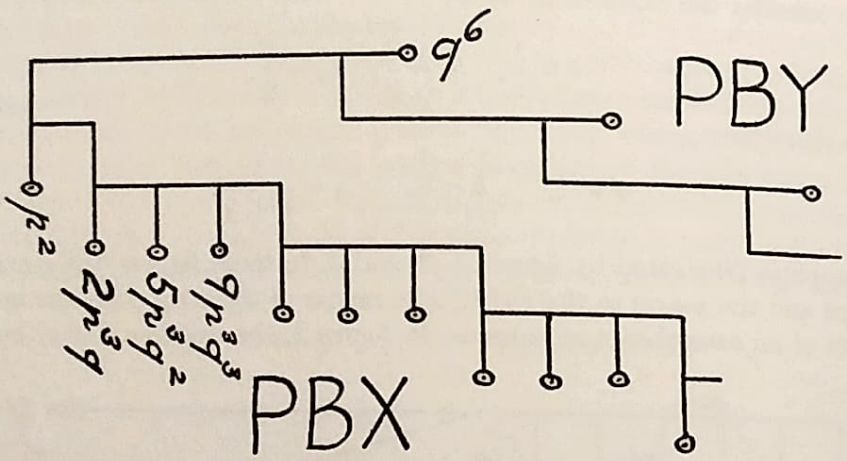

Fig. 3. The passive boundary.

2. **Computing K(x,y).** If all three points (x,y), $(x-1,y)$ and $(x,y-1)$ belong to S, then obviously

$$K(x,y) = K(x-1,y) + K(x,y-1)\,. \tag{2.1}$$

In order to give more generality to (2.1), we *extend the definition of* K(x,y) *to the active boundary of* S (but disregard the passive boundary). We set

$$K(x,y) = 0 \quad \text{if} \quad (x,y) \quad \text{in} \quad AB\,. \tag{2.2}$$

Obviously

$$K(x,y) = 1 \quad \text{if} \quad (x,y) \quad \text{in} \quad S \quad \text{and} \quad xy = 0\,; \tag{2.3}$$

the latter condition means that (x,y) lies on one of the coördinate axes. Now (2.1) holds generally for any point of S not mentioned under (2.3). In fact, (2.1) constitutes a partial difference equation for $K(x,y)$ which we have to solve under the boundary conditions (2.2) and (2.3). There is no difficulty (except boredom) in computing $K(x,y)$ in any given part of S, for instance in the initial segment S_0 (see fig. 2). We wish to discover, however, some general feature in the distribution of the values $K(x,y)$.

We consider a point (x,y) of S not contained in S_0. We use the symbol E of the calculus of finite differences to denote the displacement carrying (x,y) into $(x+a, y+b)$, and so we set

$$E\,K(x,y) = K(x+a, y+b)\,. \tag{2.4}$$

Observe that the points (x,y) and $(x+a, y+b)$ are correspondingly placed in their respective segments. This remark is essential.

In the case of figure 1, we set

$$u = K(0,2), \qquad v = K(1,1)\,.$$

As $a = 1$ and $b = 3$, we obtain

$$\begin{aligned} E\,u &= K(1,5)\,, & E\,v &= K(2,4)\,,\\ E^2u &= K(2,8)\,, & E^2v &= K(3,7)\,,\\ &\cdots & &\cdots \end{aligned}$$

Starting from the values u and v, using the difference equation (2.1) with the condition (2.2), and proceeding step by step, we can derive the values $K(x,y)$ in all points of S_1, and finally the values $E\,u$ and $E\,v$, as is visible from the following table:

	u	u	u	u	
v	$u+v$	$2u+v$	$3u+v$	$4u+v$	$= E\,u$
v	$u+2v$	$3u+3v$	$6u+4v$	$= E\,v$	

(In numerical computation, it is more advantageous to consider two tables containing merely numbers instead of the preceding one that contains also the letters u and v. Setting first $u = 1, v = 0$ and then $u = 0, v = 1$, we have the tables

	1	1	1	1
0	1	2	3	**4**
0	1	3	**6**	

	0	0	0	0
1	1	1	1	**1**
1	2	3	**4**	

Observe that these tables are, in fact, sections of Pascal's triangle and contain only binomial coefficients.)

Let us emphasize the result

$$\begin{aligned} E\,u &= 4u+v\\ E\,v &= 6u+4v \end{aligned} \tag{2.5}$$

(The numerical coefficients are obtainable from the numerical tables; see the entries in **heavy print.**)

The points in which $K(x,y)$ takes the values u and v, respectively, belong to the segment S_1, and they are *exactly so located* in S_1 as those points are in S_2 in which $K(x,y)$ takes the values $E\,u$ and $E\,v$, or as the points are in S_3 in which $K(x,y)$ takes E^2u and E^2v. Therefore, we can derive E^2u and E^2v in exactly the same manner from $E\,u$ and $E\,v$ as we have derived these latter from u and v in (2.5). Therefore, along with (2.5), we also have

$$\begin{aligned} E^2u &= 4\,E\,u + E\,v \\ E^2v &= 6\,E\,u + 4\,E\,v\,. \end{aligned} \tag{2.6}$$

Observe that we could have obtained (2.6) from (2.5) by a formal operation, multiplying (2.5) by E from the left-hand side.

By operating formally, as usual in the calculus of finite differences, we obtain from (2.5)

$$\begin{aligned} (4-E)u + \quad v \quad &= 0 \\ 6u \quad + (4-E)v &= 0\,. \end{aligned} \tag{2.7}$$

Hence, eliminating first v and then u (by multiplying the equations with suitable minors of the determinant and adding) we obtain

$$\begin{vmatrix} 4-E & 1 \\ 6 & 4-E \end{vmatrix} u = 0\,, \qquad \begin{vmatrix} 4-E & 1 \\ 6 & 4-E \end{vmatrix} v = 0\,. \tag{2.8}$$

By evaluating the determinant, we can write these equations also in the form

$$E^2u - 8\,E\,u + 10\,u = 0, \qquad E^2v - 8\,E\,v + 10\,v = 0\,. \tag{2.9}$$

Let us now consider any value w taken by $K(x,y)$ in a point of S outside S_0. This value w depends on u and v (is a certain homogeneous linear expression in u and v) and $E\,w$ *depends in the same way* on $E\,u$ and $E\,v$. For instance, in figure 1,

$$\begin{aligned} w &= K(1,3) \\ w &= 2u + v \\ E\,w &= 2E\,u + E\,v \\ E^2w &= 2E^2u + E^2v \\ &\cdots \end{aligned}$$

Thus, a suitable linear combination of the equations (2.9) yields

$$E^2w - 8\,E\,w + 10\,w = 0\,. \tag{2.10}$$

This means, written more explicitly, that

$$K(x+2, y+6) - 8\,K(x+1, y+3) + 10\,K(x,y) = 0\,. \tag{2.11}$$

There is no particular virtue in the example chosen, and our reasoning remains valid provided that s is rational. *Being given three positive numbers,* h_1, h_2, *and* s, *where* s *is rational and less than* 1, *a recursive relation of the form*

$$c_0K(x+ra, y+rb) + c_1K(x+(r-1)a, y+(r-1)b) + \cdots + c_rK(x,y) = 0 \tag{2.12}$$

can be found. The numbers r, c_0, c_1, c_2, $\cdots$, c_r *are integers,* $r \geqq 1$, $c_0 = 1$, *and depend only on* h_1, h_2, *and* s. *The equation* (2.12) *holds in any point* (x,y) *of* S *that does not belong to* S_0.

In our example the order of the recursion $r = 2$, and there are $r = 2$ points on the "bottleneck" in figure 1, marked by a slanting dotted line. Also this relation extends to the general case.

3. **A problem about random walk.** In the present section (in opposition to the foregoing section) we *extend the definition of* K(x,y) *to the passive boundary of* S (but disregard the active boundary). A point (x,y) of the passive boundary can be attained by a path that starts in (0,0), passes only through points of the strip S and ends in (x,y). The number of different paths of this kind is denoted by $K(x,y)$. Therefore

$$K(x,y) = K(x-1,y) \quad \text{if} \quad (x,y) \quad \text{in} \quad PBX, \tag{3.1}$$

$$K(x,y) = K(x,y-1) \quad \text{if} \quad (x,y) \quad \text{in} \quad PBY. \tag{3.2}$$

We consider the lines

$$x = m, \qquad y = n$$

$m = 0, 1, 2, 3, \cdots$; we call them "streets" and their intersections "corners." These streets divide the positive quarterplane into equal squares or "blocks." A hiker starts from the corner (0,0) and walks either in the positive x-direction (southward in fig. 1) or in the positive y-direction (eastward in fig. 1). The probability that, at any corner, he chooses the x-direction (south) is p, and that he chooses the y-direction (east) is q. Therefore

$$p + q = 1. \tag{3.3}$$

If, however, the hiker once leaves the strip S, he ceases to walk at random, and never returns to S; at any rate, we are no more interested in him.

The probability that the hiker attains the point (x,y), belonging to S or to its passive boundary, is $K(x,y)p^xq^y$. (These probabilities are marked in figure 3 for a few boundary points.)

The probability that the hiker attains the passive boundary in the x-direction (leaves the strip S in fig. 1 marching southward) is

$$\sum_{(x,y) \text{ in } PBX} K(x,y)p^xq^y, \tag{3.4}$$

The probability that he leaves S in the other direction (westward) is

$$\sum_{(x,y) \text{ in } PBY} K(x,y)p^x q^q . \tag{3.5}$$

The expected value of the length of his walk from (0,0) to the boundary point through which he leaves S is

$$\bar{n} = \sum_{(x,y) \text{ in } PB} (x+y)K(x,y)p^x q^y . \tag{3.6}$$

More generally, we may consider the expected value of the k-th power of the just mentioned random length,

$$\overline{n^k} = \sum_{(x,y) \text{ in } PB} (x+y)^k K(x,y)p^x q^y . \tag{3.7}$$

Using the numbers of figure 2, we may write down a few terms of these series, and there is no difficulty in principle in extending the table given in figure 2 or in writing down any desired number of terms. In possession of the results of the foregoing section, however, we can do more, namely *sum* the series in question. In fact, we set, using the numbers occurring in the recursion formula (2.12),

$$1 + c_1 p^a q^b + c_2 p^{2a} q^{2b} + \cdots c_r p^{ra} q^{rb} = R \tag{3.8}$$

and obtain

$$\sum_{(x,y) \text{ in } PBX} K(x,y)p^x q^y = \frac{P}{R} \tag{3.9}$$

$$\sum_{(x,y) \text{ in } PBY} K(x,y)p^x q^y = \frac{Q}{R} \tag{3.10}$$

where P *and* Q *are polynomials in* p *and* q.

It will suffice to show how these results are obtained in our example, characterized by (1.8). The particular cases of the foregoing formulas which refer to our example will bear the same number as the corresponding general formula but will be distinguished by an added "ex."

Thus, we obtain from figures 2 and 3

$$\sum_{(x,y) \text{ in } PBY} K(x,y)p^x q^y \tag{3.5 ex}$$

$$= q^6 + 6pq^9 + 38p^2q^{12} + \cdots$$

$$= \sum_{m=0}^{\infty} K(m,3m+6)p^m q^{3m+6} .$$

According to (2.11), $r = 2$ and

$$R = 1 - 8pq^3 + 10p^2q^6 . \tag{3.8 ex}$$

We multiply (3.5 ex) by (3.8 ex), use (2.11), and obtain:

$$\begin{aligned}
&R\sum_{(x,y) \text{ in } PBY} K(x,y)p^x q^y \\
&\quad = q^6 - 8pq^3 \cdot q^6 + 6pq^9 \\
&\quad + \sum_{m=2}^{\infty} [K(m,3m+6) - 8K(m-1,3m+3) + 10K(m-2,3m)]p^m q^{3m+6} \\
&\quad = q^6 - 2pq^9 \\
&\quad = Q
\end{aligned}$$

which proves (3.10) for our example. We prove (3.9) in a similar manner. We group the terms on the lower side of figure 3 so that power series in pq^3 appear:

(3.4 ex)
$$\begin{aligned}
&\sum_{(x,y) \text{ in } PBX} K(x,y)p^x q^y \\
&\quad = p^2 \\
&\quad + p^3q(2 + 14pq^3 + 92(pq^3)^2 + 596(pq^3)^3 + \cdots) \\
&\quad + p^3q^2(5 + 34pq^3 + 222(pq^3)^2 + \cdots) \\
&\quad + p^3q^3(9 + 60pq^3 + 390(pq^3)^2 + \cdots)
\end{aligned}$$

The recursive formula (2.11) holds between any three consecutive coefficients of these power series. Thus, we obtain by multiplying (3.4 ex) by (3.8 ex)

$$\begin{aligned}
&R\sum_{(x,y) \text{ in } PBX} K(x,y)p^x q^y \\
&\quad = p^2(1 - 8pq^3 + 10p^2q^6) \\
&\quad + p^3q(2 - 2pq^3) \\
&\quad + p^3q^2(5 - 6pq^3) \\
&\quad + p^3q^3(9 - 12pq^3) \\
&\quad = P
\end{aligned}$$

a polynomial in p and q, which proves (3.9) in our case.

It remains to calculate the expected value (3.6) or more generally (3.7). We change our standpoint for a moment: We disregard the equation (3.3) and regard p and q as independent variables. The series on the left-hand side of (3.9) and (3.10) remain convergent for sufficiently small values of p and q. Observe that R, defined by

(3.8), is a polynomial in p and q. The calculation that we have just carried through in our example shows that the formulas (3.9) and (3.10) remain valid, P and Q being well-defined polynomials in p and q. It follows that

$$\sum_{(x,y) \text{ in } PB} K(x,y)p^x q^y = \frac{P+Q}{R}. \tag{3.11}$$

The right-hand side is a rational function of the independent variables p and q which takes the value 1 if the condition (3.3) is fulfilled.[3]

Yet we regard p and q still as independent and consider the differential operation

$$D = p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}. \tag{3.12}$$

If α and β denote constants and U and V functions of p and q,

$$Dp^\alpha q^\beta = (\alpha+\beta)p^\alpha q^\beta$$

$$D(\alpha U + \beta V) = \alpha DU + \beta DV, \qquad DUV = UDV + VDU, \tag{3.13}$$

$$D\frac{U}{V} = \frac{VDU - UDV}{V^2}.$$

Thus we obtain from (3.11)

$$\sum_{(x,y) \text{ in } PB} (x+y)^k K(x,y)p^x q^y = D^k \frac{P+Q}{R}. \tag{3.14}$$

If, *after* differentiation, we reintroduce the condition (3.3), we obtain from (3.7) $\overline{n^k}$ as a rational function of p. Observing that the right-hand side of (3.11) is = 1 if (3.3) holds, we obtain in particular

$$\begin{aligned} \bar{n} &= \frac{DP + DQ - DR}{R}, \\ &= \frac{1}{R}\frac{\partial}{\partial p}(P+Q-R) \\ &= \frac{1}{R}\frac{\partial}{\partial q}(P+Q-R) \end{aligned} \tag{3.15}$$

where, after differentiation, the relation (3.3) must be taken into account.[4]

[3] This must, of course, be proved. See [3] p. 134 or [2] p. 14.

[4] A summary of the present paper was presented to the Paris Academy on June 17, 1946, under the title: Sur une généralisation d'un problème élémentaire classique, importante dans l'inspection des produits industriels. See Comptes Rendus, vol. 222 (1946) pp. 1422–1424. [Added in proof, January 27, 1948.]

Stanford University

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